Indian Statistical Institute
Mid-Semestral Examination
Differential Geometry MMath I

Max Marks: 40 Time: 3 hours

(1) Show that if the Gauss map of a compact connected n-surface is one-one, then the surface is homeomorphic to the sphere S^n. [4]

(2) Give examples of three non-zero vector fields on $\mathbb{R}^3$ such that all of them restrict to a tangent vector fields on S^2. For any one of these vector field restricted to the sphere S^2, find the integral curve through the point $(-1, 0, 0)$. [5]

(3) Let (a_{ij}) be a real symmetric $(n+1)\times(n+1)$ matrix. Show that the maximum and minimum values of the function

$$f(X) = \sum_{i,j} a_{ij}x_i x_j$$

where $X = (x_1, \ldots, x_{n+1}) \in S^n$ are eigenvalues of the matrix (a_{ij}). [4]

(4) Show that if $\alpha : I \longrightarrow S$ is a geodesic in an n-surface S and if $\beta = \alpha \circ h$ is a reparametrization of α, where $h : J \longrightarrow I$ is a surjective map with $h' > 0$, then β is a geodesic if and only there exists $a, b \in \mathbb{R}$ with $a > 0$ such that $h(t) = at + b$ for all $t \in J$. [5]

(5) Let C be a plane curve in the upper half-plane $(x_2 > 0)$ and let S be the surface of revolution obtained by rotating C about the x_1-axis. Let $\alpha(t) = (x_1, x_2)$ be a constant speed parametrized curve in C. For $t, \theta \in \mathbb{R}$ define

$$\alpha_\theta(t) = (x_1(t), x_2(t)\cos\theta, x_2(t)\sin\theta)$$
$$\beta_t(\theta) = (x_1(t), x_2(t)\cos\theta, x_2(t)\sin\theta)$$

Show that (i) each α_θ is a geodesic in S and (ii) β_t is a geodesic if and only if the slope of the tangent line to C at $\alpha(t)$ is zero. [6]

(6) Let S be a 2-surface in $\mathbb{R}^3$ and let $\alpha : I \longrightarrow S$ be a geodesic in S with $\dot{\alpha} \neq 0$. Show that a vector field $\mathbb{X}$ tangent to S along α is parallel if and only if both $||\mathbb{X}||$ and the angle between $\mathbb{X}$ and $\dot{\alpha}$ are constant. [6]

(7) For $\theta \in \mathbb{R}$, let $\alpha_\theta : [0, \pi] \longrightarrow S^2$ be the curve defined by

$$\alpha_\theta(t) = (\cos\theta \sin t, \sin\theta \sin t, \cos t).$$

Let $p = (0, 0, 1)$ and $v = (p, 1, 0, 0) \in S^2_p$. Compute $P_{\alpha_\theta}(v)$. [5]

(8) Discuss the definition of the Weingarten map. Compute the Weingarten map for (i) the hyperplane $a_1x_1 + \cdots + a_{n+1}x_{n+1} = 0$ where not all a_i are zero and (ii) the cylinder $x_2^2 + x_3^2 = a^2$, $a \neq 0$. [5]